Use of the Centrifuge in Particle Size Analysis – Equation Theory and Derivation
(J.C. Hathaway, Written Communication)

Theory

$W = ma_r$

Where: W = apparent weight of a spherical particle
m = mass of the particle
a_r = radial acceleration

$a_r = w^2R$

Where: w = angular velocity in radians/sec
R = distance of particle from center of rotation

$W = mw^2R$

$m = \rho V$

Where: ρ = density of the particle in grams/cm^3
V = volume of the particle in cm^3

$V = 3/4\pi r^3$

Where: r = radius of the particle in cm

Therefore,

$W = 4/3\ \pi r^3 \rho w^2 R$

$F = W - B$

Where: F = net force acting on the particle
B = buoyant force = $4/3\ \pi r^3 \rho_o w^2 R$
Where: ρ_o = density of displaced medium

$F = 4/3\ \pi r^3 \rho w^2 R - 4/3\ \pi r^3 \rho_o w^2 R$

Therefore,

$F = 4/3\ \pi r^3 w^2 R(\rho - \rho_o)$

By Stokes Law

$F = 6\pi\eta r v$

Where: η = viscosity of the medium in poises (dyne-sec/cm^2)
r = radius of the particle in cm
v = velocity of the particle in cm/sec

$6\pi\eta r v = 4/3\ \pi r^3 w^2 R(\rho - \rho_o)$

Solving for v

$v = 4\pi r^3 w^2 R(\rho - \rho_o) / 3(6\pi\eta r) = 2r^2 w^2 R(\rho - \rho_o) / 9\eta$

$v = ds/dt$

Where: s = distance = R
t = time

$$ds = dR$$

$$v = dR/dt$$

Therefore:

$$dR / dt = 2r^2w^2R(\rho-\rho_o) / 9\eta$$

Inverting:

$$dt / dR = 9\eta / 2r^2w^2R(\rho-\rho_o)$$

$$t = \int_{R_1}^{R_2} 9\eta / 2r^2w^2R(\rho-\rho_o)\, dR$$

$$t = 9\eta / 2r^2w^2R(\rho-\rho_o) \int_{R_1}^{R_2} dR/R$$

Integrating:

$$t = 9\eta / 2r^2w^2(\rho-\rho_o) [\ln R_2 - \ln R_1]$$

$$t = 9\eta \ln(R_2/R_1) / 2r^2w^2(\rho-\rho_o)$$

To allow for acceleration and deceleration of the centrifuge:

$$w = w_o + \alpha t$$

Where: α = angular acceleration

$$dt/dR = 9\eta / 2r^2R(\rho-\rho_o) (w_o + \alpha t)^2$$

$$dt/dR = 9\eta / 2r^2R(\rho-\rho_o) (w_o^2 + 2w_o\alpha t + \alpha^2t^2)$$

$$\int_{t_o}^{t_a} (w_o^2 + 2w_o\alpha t + \alpha^2t^2)\, dt = 9\eta / 2r^2(\rho-\rho_o) \int_{R_1}^{R_a} dR/R$$

$$w_o^2\int_{t_o}^{t_a} dt + 2w_o\alpha\int_{t_o}^{t_a} t\, dt + \alpha^2\int_{t_o}^{t_a} t^2\, dt = 9\eta / 2r^2(\rho-\rho_o) \int_{R_1} dR/R$$

$$t_o = 0$$

$$w_o^2 ta + 2w_o\alpha t_a^2/2 + \alpha^2 t_a^3/3 = 9\eta (\ln R_a - \ln R_1) / 2r^2(\rho-\rho_o)$$

$w_o = 0 \qquad \alpha = \alpha_1 = \text{constant (assumed)}$

$\alpha_1^2 t_a^3/3 = 9\eta\,(\ln R_a - \ln R_1)/\,2r^2(\rho-\rho_o)$

$\alpha_1^2 t_a^3 = w_1^2$

$w_1 = w_o + \alpha_1 t_a = \alpha_1 t_a$

$w_1^2 t_a/3 = 9\eta\,(\ln R_a - \ln R_1)/\,2r^2(\rho-\rho_o)$

$2r^2(\rho-\rho_o)w_1^2 t_a/3(9\eta) = \ln R_a - \ln R_1$

$\ln R_a = [2r^2(\rho-\rho_o)w_1^2 t_a/3(9\eta)] + \ln R_1$

At t_a, operating velocity is reached and particle is at distance Ra from axis of rotation.

$w_1 = w_o = \text{constant} \quad \alpha = 0$

$$\int_{t_a}^{t_b} (w_1^2 + 2w_1\alpha t + \alpha^2 t^2)\,dt = 9\eta\,/\,2r^2(\rho-\rho_o) \int_{R_a}^{R_b} dR/R$$

Since $\alpha = 0$

$w_1^2(t_b - t_a) = 9\eta\,(\ln R_b - \ln R_a)/\,2r^2(\rho-\rho_o)$

$t_b - t_a = 9\eta\,[\ln R_b - (2r^2(\rho-\rho_o)w_1^2 t_a/3(9\eta)) + \ln R_1]\,/\,2r^2\,w_1^2(\rho-\rho_o)$

$t_b - t_a = 9\eta\,[\ln R_b + \ln R_1 - (2r^2 w_1^2 t_a\,(\rho-\rho_o)/3(9\eta))]\,/\,2r^2\,w_1^2(\rho-\rho_o)$

$t_b - t_a = \{9\eta\,[\ln R_b + \ln R_1]\,/\,2r^2\,w_1^2(\rho-\rho_o)\} - \{9\eta\,(2r^2 w_1^2 t_a\,(\rho-\rho_o)\,/\,3(9\eta)2r^2\,w_1^2(\rho-\rho_o)\}$

$t_b - t_a = \{9\eta\,[\ln R_b + \ln R_1]\,/\,2r^2\,w_1^2(\rho-\rho_o)\} - t_a\,/\,3 = t_R$

At t_b, particle has reached distance R_b from axis of rotation and deceleration is begun.

$w_o = w_1 \qquad w_f = \text{final velocity} = 0 \qquad \alpha = \alpha_2$

$w_f = w_1 - \alpha(t_c - t_b)$

$w_1 = \alpha_2(t_c - t_b)$

$dt\,/\,dR = 9\eta\,/\,2r^2 R(\rho-\rho_o)(w_1 - \alpha_2 t)^2$

$dt / dR = 9\eta / 2r^2R(\rho-\rho_o)(w_1^2 - 2w_1\alpha_2 t+\alpha_2^2t^2)$

$$\int_{t_b}^{t_a} (w_1^2 - 2w_1\alpha_2 t+\alpha_2^2t^2)\, dt = 9\eta / 2r^2(\rho-\rho_o)\int_{R_b}^{R_2} dR/R$$

$w_1^2 (t_c - t_b) - (2w_1\alpha_2 (t_c^2 - t_b^2) / 2) + (\alpha_2^2 (t_c^3 - t_b^3) / 3) = 9\eta(\ln R_2\text{-}\ln R_b) / 2r^2(\rho-\rho_o)$

Since deceleration stage is begun at t_b, assume $t_b = 0$ with $t_c - t_b = t_d$

Then:
$w_1^2 t_d - w_1\alpha_2 t_d^2 + \alpha_2^2 t_d^3 / 3 = 9\eta(\ln R_2\text{-}\ln R_b) / 2r^2(\rho-\rho_o)$

$w_1 = \alpha_2 t_d$

$w_1^2 t_d - w_1^2 t_d + w_1^2 t_d / 3 = 9\eta(\ln R_2\text{-}\ln R_b) / 2r^2(\rho-\rho_o)$

$\ln R_b = \ln R2 - 2r^2w_1^2 t_d (\rho-\rho_o) / 3(9\eta)$

$t_r = 9\eta[\ln R_b\text{-}\ln R_1] / 2r^2 w_1^2 (\rho-\rho_o) - t_a/3$

Where: t_r = running time = time at constant velocity

$t_r = \{9\eta[(\ln R_2\text{-} (2r^2 w_1^2 t_d(\rho-\rho_o)) / 3(9\eta)) - \ln R_1] / 2r^2 w_1^2 (\rho-\rho_o)\} - t_a/3$

$t_r = \{9\eta[(\ln R_2\text{-} \ln R_1] / 2r^2 w_1^2 (\rho-\rho_o) - (9\eta)2r^2 w_1^2 t_d(\rho-\rho_o) / 3(9\eta)2r^2 w_1^2(\rho-\rho_o)\} - t_a/3$

$t_r = \{9\eta[(\ln R_2\text{-} \ln R_1] / 2r^2 w_1^2 (\rho-\rho_o)\} - (t_d/3 + t_a/3)$

$t_r = \{9\eta[(\ln R_2\text{-} \ln R_1] / 2r^2 w_1^2 (\rho-\rho_o)\} - (t_d + t_a)/3$

$T = t_a + t_r + t_d$

Where: T = total time

$T = t_a + \{[9\eta \ln(R_2/R_{1)} / 2r^2 w_1^2 (\rho-\rho_o)] - (t_a + t_d)/3\} + t_d$

$T = [9\eta \ln(R_2/R_{1)} / 2r^2 w_1^2 (\rho-\rho_o)] + (3t_a/3 + 3t_d/3) - (t_a + t_d)/3$

$T = [9\eta \ln(R_2/R_{1)} / 2r^2 w_1^2 (\rho-\rho_o)] + [3(t_a + t_d) - (t_a + t_d)]/3$

$T = [9\eta \ln(R_2/R_{1)} / 2r^2 w_1^2 (\rho-\rho_o)] + [2(t_a + t_d)]/3$

$w_1 = 2\pi N$

Where: N = angular velocity in revolutions/sec

$T = [9\eta \ln(R_2/R_1) / (2)4\pi^2 N^2 r^2(\rho - \rho_o)] + 2(t_a + t_d)/3$

$\ln(R_2/R_1) = \log_{10}(R_2/R_1)/\log_{10}e = \log_{10}(R_2/R_1) / .434294481$

$T = [9\eta \log_{10}(R_2/R_1) / (2)(4)(.434)\pi^2 N^2 r^2(\rho - \rho_o)] + 2(t_a + t_d)/3$

$T = [\eta \log_{10}(R_2/R_1) / (3.81)N^2 r^2(\rho - \rho_o)] + 2(t_a + t_d)/3$

or

$T = [(9)\ln(R_2/R_1)(\eta/\rho - \rho_o)]/ 2(4)\pi^2(N/60)^2 ((d/2)\ 10^{-4})^2] + 2(t_a + t_d)/3$

Where:

T	=	total time (sec)
t_a	=	time of acceleration (sec)*
t_r	=	time at constant velocity (sec)
t_d	=	time of deceleration (sec)*
η	=	viscosity (poises)
R_1	=	initial distance from axis of rotation (cm)
R_2	=	final distance from axis of rotation (cm)
r	=	radius of particle (cm)
N	=	angular velocity (rev/sec)
ρ	=	density of particle (g/cm^3)
ρ_o	=	density of medium (g/cm^3)

* Acceleration and deceleration are assumed to be constant